Minimal renormalization without ϵ-expansion: Three-loop amplitude functions of the O(n) symmetric ϕ^4 theory in three dimensions below T_c

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Abstract

We present an analytic three-loop calculation for thermodynamic quantities of the O(n) symmetric ϕ^4 theory below T_c within the minimal subtraction scheme at fixed dimension $d = 3$. Goldstone singularities arising at an intermediate stage in the calculation of O(n) symmetric quantities cancel among themselves leaving a finite result in the limit of zero external field. From the free energy we calculate the three-loop terms of the amplitude functions f_ϕ, F_+ and F_- of the order parameter and the specific heat above and below T_c, respectively, without using the $\epsilon = 4-d$ expansion. A Borel resummation for the case $n = 2$ yields resummed amplitude functions f_ϕ and F_- that are slightly larger than the one-loop results. Accurate knowledge of these functions is needed for testing the renormalization-group prediction of critical-point universality along the λ-line of superfluid ^{4}He. Combining the three-loop result for F_- with a recent five-loop calculation of the additive renormalization constant of the specific heat yields excellent agreement between the calculated and measured universal amplitude ratio A^+/A^- of the specific heat of ^{4}He. In addition we use our result for f_ϕ to calculate the universal combination R_C of the amplitudes of the order parameter, the susceptibility and the specific heat for $n = 2$ and $n = 3$. Our Borel-resummed three-loop result for R_C is significantly more accurate than the previous result obtained from the ϵ-expansion up to $O(\epsilon^2)$.

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1 Introduction

Field-theoretic perturbative calculations of the critical behavior of O(n) symmetric systems with $n > 1$ below T_c are known to be considerably more complicated than those of Ising-like ($n = 1$) systems. The difficulties for $n > 1$ are due to the existence of transverse fluctuations of the order parameter (in addition to the ordinary longitudinal fluctuations of $n = 1$ systems) which, in the long-wavelength limit and at vanishing external field, have a vanishing restoring force. This implies the existence of massless Goldstone modes [1,2] which yield infinite transverse and longitudinal susceptibilities [3] and which cause infrared (Goldstone) singularities at intermediate stages of perturbative calculations of all thermodynamic quantities on the coexistence curve below T_c. These complications for $n > 1$ have prevented calculations of the equation of state and of amplitude functions below T_c to higher than two-loop order [4–6] , in contrast to the case $n = 1$ below T_c where accurate Borel resummation results based on five-loop perturbation theory are available [7–10].

Higher-order calculations of the amplitude functions below T_c for the case $n = 2$ are of primary importance in view of the proposed theoretical research [11] parallel to the considerable experimental effort [12] to test the universality prediction of the renormalization-group theory [13] along the λ-line of ^{4}He. The amplitude functions contain the information about universal ratios [13] of leading and subleading amplitudes near criticality that have previously been used as fit parameters in the data analysis [14,15] because of the lack of accurate predictions for these universal ratios. The amplitude functions are also a crucial ingredient in a nonlinear renormalization-group analysis [16–18] (equivalent to a resummation of the whole Wegner correction [19] series) in a wide temperature range including non-asymptotic and non-universal effects.

In the present paper we perform the next necessary step towards the goal of an accurate determination of amplitude functions by presenting the analytic results of a three-loop calculation within the O(n) symmetric ϕ^4 theory above and below T_c for general n. Specifically, from the free energy we shall derive the amplitude function f_ϕ of the order parameter and the amplitude functions $F_\pm$ of the specific heat above and below T_c in the limit of vanishing external field. The conceptual framework of our calculation is the minimally renormalized massive ϕ^4 field theory at fixed dimension $d = 3$ [5,17,20], without using the $\epsilon = 4 - d$ expansion, involving an appropriately defined pseudo-correlation legth ξ_- [5,6] that is applicable to both $n = 1$ and $n > 1$ below T_c. Our $d = 3$ approach differs from Parisi's [21] original suggestion of the $d = 3$ field theory and from the $d = 3$ field theory of subsequent work [7,9,22–24] where renormalization conditions rather than the minimal renormalization scheme were used. Our combination of the minimal subtraction scheme with the field theory at fixed $2 < d < 4$ has several advantages, such as the independence

of renormalization constants on whether $T > T_c$ or $T < T_c$ which implies a natural decomposition of correlation functions into amplitude functions and exponential parts where the structure of the latter is independent of whether $T > T_c$ or $T < T_c$. The same exponential parts can then be used, without modification, in extensions of the theory to critical dynamics [25] or to critical phenomena in confined systems above and below T_c [26].

Our approach has been successfully employed recently [6] in deriving various amplitude functions, including f_ϕ and $F_\pm$, for general n below T_c in two-loop order. In three-loop order, the technical difficulties related to the removal of ultraviolet divergences in three dimensions and to the treatment of spurious infrared (Goldstone) divergences are substantially greater. The concept of the minimally renormalized $d = 3$ field theory turns out to constitute an appropriate framework for coping with these difficulties. This requires to further develop new integration techniques based on recent advances by Rajantie [27] in the analytic evaluation of three-loop integrals in three dimensions. In particular we succeed in calculating two "Mercedes" diagrams with two different (longitudinal and transverse) masses that were not considered previously.

The perturbation series of the ϕ^4 theory are known to be divergent and to require resummations [28] in order to yield reliable results. The three-loop terms derived in the present paper make possible to perform Borel resummations that lead to results with reasonably small error bars. In a related paper [10] this was shown for F_-. In the present paper we demonstrate this for the case of f_ϕ. For the example $n = 2$ the Borel resummed three-loop amplitude functions F_- and f_ϕ of the specific heat below T_c and of the order parameter turn out to be slightly larger than the one-loop results. Application of the Borel resummed amplitude function F_- to the universal amplitude ratio A^+/A^- of the asymptotic specific heat yields the theoretical prediction [10] $A^+/A^- = 1.056 \pm 0.004$ which is in excellent agreement with the high-precision experimental result [15] $A^+/A^- = 1.054 \pm 0.001$ for ^{4}He near the superfluid transition obtained from a recent experiment in space. Furthermore we use our result for f_ϕ to calculate the universal combination R_C [13] of the amplitudes of the order parameter, the susceptibility and the specific heat for $n = 2$ and $n = 3$. Our Borel resummed three-loop result for R_C is significantly more accurate than the previous result obtained from the ϵ-expansion up to $O(\epsilon^2)$ [29].

2 Bare Helmholtz free energy

Throughout this paper we shall use the notation and definitions of Ref. [6]. We start from the O(n) symmetric ϕ^4 model with the Landau-Ginzburg-Wilson

functional

$$\mathcal{H}\{\vec{\phi}_0(\mathbf{x})\} = \int\limits_V \mathrm{d}^d x \left(\frac{1}{2} r_0 \phi_0^2 + \frac{1}{2} \sum_i (\nabla \phi_{0i})^2 + u_0 (\phi_0^2)^2 - \vec{h}_0 \cdot \vec{\phi}_0 \right) \tag{1}$$

for the n-component field $\vec{\phi}_0(\mathbf{x}) = (\phi_{01}(\mathbf{x}), \ldots, \phi_{0n}(\mathbf{x}))$ in the presence of the homogeneous external field $\vec{h}_0 = (h_0, 0, \ldots, 0)$. The spatial variations of $\vec{\phi}_0(\mathbf{x})$ are restricted to wavenumbers less than some cutoff Λ. We are interested in the bulk Helmholtz free energy $\Gamma_0(r_0, u_0, M_0, \Lambda)$ per unit volume with $M_0 \equiv \langle \phi_{01} \rangle$. Γ_0 is obtained from the negative sum of all one-particle irreducible vacuum diagrams. The structure of the analytic expression is (apart from an unimportant additive constant)

$$\begin{aligned}\Gamma_0(r_0, u_0, M_0, \Lambda) = {} & \frac{1}{2} r_0 M_0^2 + u_0 M_0^4 + \frac{1}{2} \int\limits_{\mathbf{p}}^{\Lambda} \ln \left(\bar{r}_{0L} + p^2 \right) \\ & + \frac{1}{2}(n-1) \int\limits_{\mathbf{p}}^{\Lambda} \ln \left(\bar{r}_{0T} + p^2 \right) + u_0 X_0^{(2)}(r_0, u_0, M_0, \Lambda) \\ & + u_0^2 X_0^{(3)}(r_0, u_0, M_0, \Lambda) + O(u_0^3) \end{aligned} \tag{2}$$

where $\int_{\mathbf{p}}^{\Lambda} \equiv (2\pi)^{-d} \int^{\Lambda} \mathrm{d}^d p$ means integration up to $|\mathbf{p}| < \Lambda$. The terms $u_0 X_0^{(2)}(r_0, u_0, M_0, \Lambda)$ and $u_0^2 X_0^{(3)}(r_0, u_0, M_0, \Lambda)$ represent the two- and three-loop contributions shown in Fig. 1, with longitudinal and transverse propagators $G_L(p) = (\bar{r}_{0L} + p^2)^{-1}$ and $G_T(p) = (\bar{r}_{0T} + p^2)^{-1}$ where

$$\bar{r}_{0L} = r_0 + 12 u_0 M_0^2 \,, \quad \bar{r}_{0T} = r_0 + 4 u_0 M_0^2 \tag{3}$$

(compare Eq. (27) and Fig. 3 of Ref. [6] at $k = 0$). For the integral expressions of $X_0^{(3)}$ see Appendix A.

We consider the limit $\Lambda \to \infty$ (although neglecting cutoff effects may not be justified in certain cases [30–32]). The right-hand side of Eq. (2) is ultraviolet divergent for $\Lambda \to \infty$ at any $d > 2$. To absorb these divergences requires two different steps: a shift of the temperature variable r_0 (mass shift) and a subtraction from the free energy. The first step is to turn to the shifted variable $r_0 - r_{0c}$ where $r_{0c}(u_0, \Lambda)$ is the critical value of r_0 [20]. Substituting $r_0 = r_0 - r_{0c} + r_{0c}(u_0, \Lambda)$ into Γ_0 yields the function

$$\hat{\Gamma}_0(r_0 - r_{0c}, u_0, M_0, \Lambda) = \Gamma_0(r_0 - r_{0c} + r_{0c}(u_0, \Lambda), u_0, M_0, \Lambda) \,. \tag{4}$$

(Because of the non-analytic u_0 dependence of r_{0c}, the function $\hat{\Gamma}_0$ does not have an expansion in integer powers of u_0.) This function is still divergent for

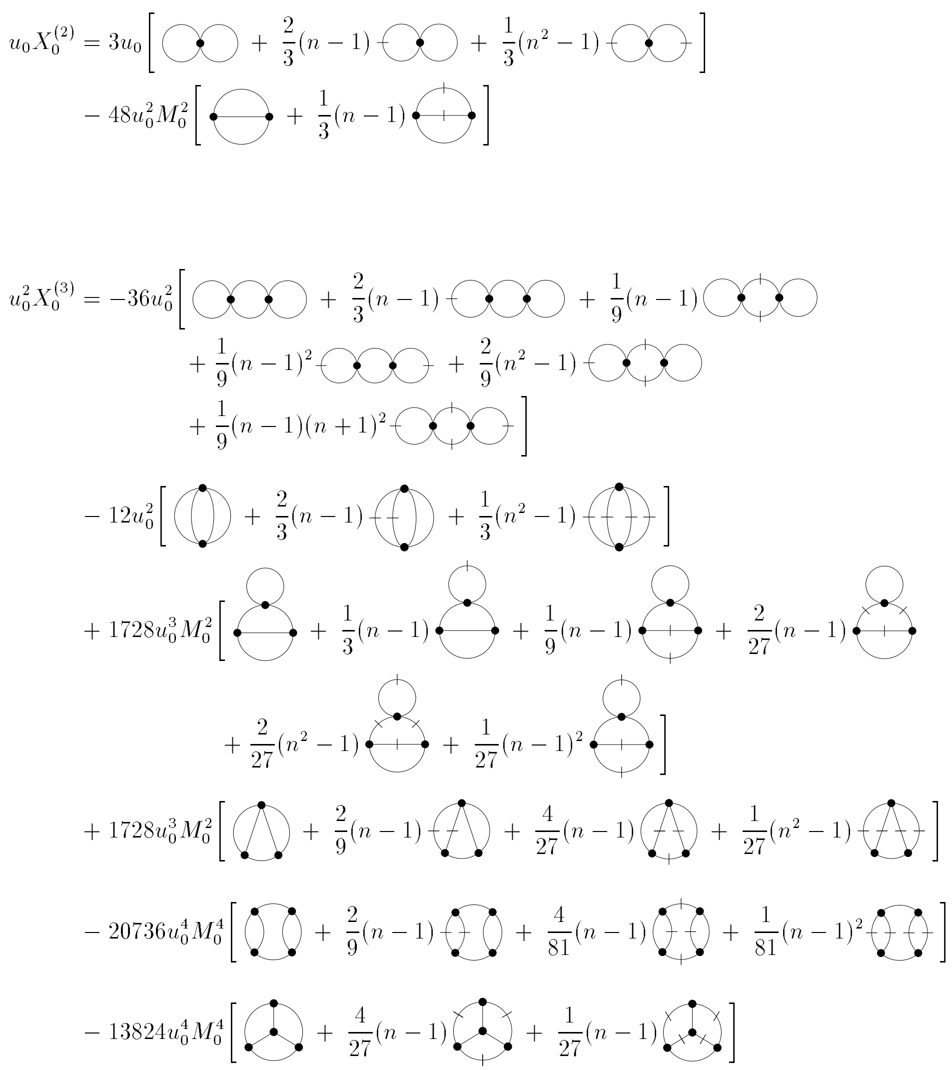

Fig. 1. Two- and three-loop vacuum diagrams determining the Helmholtz free energy $\Gamma_0(r_0, u_0, M_0, \Lambda)$, Eq. (2). The lines denote longitudinal and transverse propagators, $G_L(p) =$ ——— $\equiv (\bar{r}_{0L} + p^2)^{-1}$ and $G_T(p) =$ —+— $\equiv (\bar{r}_{0T} + p^2)^{-1}$. The integral expressions are given in Appendix A. The analytic results for the last two "Mercedes" diagrams at $d = 3$ are given in Eqs. (A.7) and (A.8).

$d > 2$ in the limit $\Lambda \to \infty$ at fixed $r_0 - r_{0c}$, u_0, M_0. We have verified up to three-loop order that these ultraviolet divergences have a regular dependence on $r_0 - r_{0c}$ (up to linear order in $r_0 - r_{0c}$ for $d < 4$ and up to quadratic order in $r_0 - r_{0c}$ for $d = 4$). Therefore it suffices to perform additional overall subtractions from $\hat{\Gamma}_0$ that are regular in $r_0 - r_{0c}$. They do not affect the singular part of the temperature dependence of the free energy. The resulting singular

part of the bare free energy is finite in the limit $\Lambda \to \infty$ for $2 < d < 4$ at fixed $r_0 - r_{0c}$.

Here we use the prescriptions of dimensional regularization at $\Lambda = \infty$ and employ the critical parameter r_{0c} in the form [20,33]

$$r_{0c}(u_0, \epsilon) = u_0^{2/\epsilon} S(\epsilon) \tag{5}$$

where the function $S(\epsilon)$ is finite for $\epsilon > 0$ except for poles at $d_l = 4 - 2/l$, $l = 2$, 3, ... In our version of the $d = 3$ theory it is sufficient to work near $d = 3$ and, instead of $r_0 - r_{0c}$, to use a shifted variable r'_0 as defined by $r_0 = r'_0 + \delta r_0(u_0, \epsilon)$ with a simplified $\delta r_0(u_0, \epsilon)$ that contains only the $d = 3$ pole of r_{0c} (and not the poles at $d_l > 3$ with $l \geq 3$). Thus we use [6–8]

$$\delta r_0(u_0, \epsilon) = \frac{1}{\pi^2}(n+2)\frac{u_0^{2/\epsilon}}{\epsilon - 1} + C(n)\, u_0^{2/\epsilon} \tag{6}$$

with $\epsilon = 4 - d$ and the finite constant

$$C(n) = \frac{n+2}{\pi^2}\left[1 - C_{\mathrm{Euler}} + \ln\frac{4\pi}{9} - 2\ln 24\right] \tag{7}$$

(the final results for the amplitude function do not depend on the choice of $C(n)$ [6,20]). Correspondingly, instead of $\bar{r}_{0L}$ and $\bar{r}_{0T}$, we use the longitudinal and transverse parameters

$$r_{0L} = r'_0 + 12u_0 M_0^2\,, \quad r_{0T} = r'_0 + 4u_0 M_0^2\,. \tag{8}$$

In the limit $d \to 3$ we then obtain the singular part of the bare Helmholtz free energy as a finite function $\mathring{\Gamma}(r'_0, u_0, M_0)$ whose two-loop expression has been derived recently (see Eq. (31) of Ref. [6]).

We have been able to perform an analytic calculation of the three-loop contributions to $\mathring{\Gamma}(r'_0, u_0, M_0)$ near the coexistence curve below T_c and for $M_0^2 = 0$ above T_c. The details of this calculation will be presented elsewhere. Some comments are given in Appendix A. The resulting bare perturbation expression for the Helmholtz free energy at $r'_0 \neq 0$ reads up to three-loop order at $d = 3$

$$\mathring{\Gamma}(r'_0, u_0, M_0) = \frac{1}{2}r'_0 M_0^2 + u_0 M_0^4 + \sum_{b=1}^{3}\sum_{l=0}^{b-1}\sum_{k=0}^{1}(-1)^k\, 2^{-l-k} F_{blk}(\bar{w}, n) \cdot (24u_0)^{3-l}(M_0^2)^l \left[\frac{r_{0L}}{(24u_0)^2}\right]^{\frac{4-b-2l}{2}} \left(\ln\frac{r_{0L}}{(24u_0)^2}\right)^k \tag{9}$$

where

$$\bar{w}(r_0', u_0, M_0) = r_{0T}/r_{0L} \tag{10}$$

is a non-perturbative parameter. Above T_c its largest possible value is $\bar{w}(r_0', u_0, 0) = 1$ (when $r_0' > 0$ and $M_0^2 = 0$ corresponding to $T > T_c$ at $h_0 = 0$). The smallest possible value of $\bar{w}$ is attained on the coexistence curve well below T_c where, at given $r_0' < 0$ and to lowest order in u_0, $\bar{w}(r_0', u_0, M_0(r_0', u_0)) = 3\pi^{-1} u_0 (-2r_0')^{-1/2} + O(u_0^2)$ as follows from Eq. (22) in Sect. 3. The coefficients $F_{blk}(\bar{w}, n)$ depend on $\bar{w}$ and n and determine the contributions in b-loop order. The one- and two-loop coefficients are [6]

$$F_{100}(\bar{w}, n) = -\frac{1}{12\pi}\left[1 + (n-1)\bar{w}^{3/2}\right], \tag{11}$$

$$F_{200}(\bar{w}, n) = \frac{1}{384\pi^2}\left[3 + 2(n-1)\bar{w}^{1/2} + (n^2-1)\bar{w}\right], \tag{12}$$

$$F_{210}(\bar{w}, n) = \frac{1}{288\pi^2}(n-1)\ln\frac{1+2\bar{w}^{1/2}}{3}, \tag{13}$$

$$F_{211}(n) = -\frac{1}{288\pi^2}(n+2). \tag{14}$$

The new three-loop coefficients F_{300} and F_{301} read in analytic form

$$\begin{aligned} F_{300}(\bar{w}, n) = \frac{1}{18432\pi^3}\Bigg\{ & 15 + 24\ln\frac{3}{4} - (n-1)\Bigg[\bar{w}^{-1/2} + 2n - 6 \\ & + 8\ln\frac{2+2\bar{w}^{1/2}}{3} + \bar{w}^{1/2}\Bigg(n^2 - 6n - 9 + 4(n+1)\ln\frac{16\bar{w}}{9} \\ & + 8\ln\frac{2+2\bar{w}^{1/2}}{3}\Bigg) + \bar{w}(n-1)\Bigg]\Bigg\}, \end{aligned} \tag{15}$$

$$F_{301}(\bar{w}, n) = \frac{1}{2304\pi^3}\left\{3 + (n-1)\left[1 + (n+2)\bar{w}^{1/2}\right]\right\}. \tag{16}$$

In calculating the coefficients F_{310} and F_{320} below T_c we have confined ourselves to the vicinity of the coexistence curve where an expansion with respect to $\bar{w}$ is justified. For the purpose of calculating the order parameter this expansion must include the terms of linear order in $\bar{w}$ whereas for the calculation of the specific heat all terms can be neglected that vanish in the limit $\bar{w} \to 0$. The latter statement applies to F_{300} and F_{301} as well. The result for the coefficients F_{310} and F_{320} is

$$F_{310}(\bar{w}, n) = \frac{1}{27648\pi^3}\Bigg\{9\pi^2 - 18 + 108\,\mathrm{Li}_2\left(-\tfrac{1}{3}\right) - (n-1)\Bigg[4\bar{w}^{-1/2}$$

$$
\begin{aligned}
&+ 4n + 2 - (n+2)\pi^2 - 12\,\mathrm{Li}_2\!\left(\tfrac{1}{3}\right) - 32\ln 2 - 6\,(\ln 3)^2 \\
&+ \bar{w}^{1/2}\Big(10n + 32 - 16\,(2n+3)\ln 2 + 48\ln 3 - 8(n+1)\ln\bar{w}\Big) \\
&+ \frac{1}{3}\bar{w}\Big(84n - 100 - 128\ln 2\Big) + O(\bar{w}^{3/2}, \bar{w}^{3/2}\ln\bar{w})\Big]\Big\}\,, \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
F_{320}(\bar{w}, n) = \frac{1}{165888\pi^3}\Big\{ & 432\ln\frac{4}{3} - 324\,\mathrm{Li}_2\!\left(-\tfrac{1}{3}\right) - 432c_1 - 27\pi^2 \\
& - (n-1)\Big[16\bar{w}^{-1/2} + \frac{3n+14}{3}\pi^2 + 18(\ln 3)^2 + 36\,\mathrm{Li}_2\!\left(\tfrac{1}{3}\right) \\
& + 16\Big(c_2 + 4\,\mathrm{Li}_2(-2) - 2\,\mathrm{Li}_2\!\left(-\tfrac{1}{2}\right) + \Big[6\ln 3 - \ln 2 - \frac{13}{3}\Big]\ln 2\Big) \\
& - \frac{128}{3} + 16\bar{w}^{1/2}\Big(7 - n + (n+1)\ln(16\bar{w}) + 2\ln 2 - 6\ln 3\Big) \\
& + 4\bar{w}\Big(4c_2 - 12n - \frac{224}{5} + \pi^2 + 6\left[6\ln 3 - \ln 2 - \frac{16}{15}\right]\ln 2 \\
& + 12\left[2\,\mathrm{Li}_2(-2) - \mathrm{Li}_2\!\left(-\tfrac{1}{2}\right)\right]\Big) + O(\bar{w}^{3/2}, \bar{w}^{3/2}\ln\bar{w})\Big]\Big\}. \qquad (18)
\end{aligned}
$$

Here $\mathrm{Li}_2(x) \equiv \int_x^0 t^{-1}\ln(1-t)\,\mathrm{d}t$ is the dilogarithmic function. The constants c_1 and c_2 are given by

$$
\begin{aligned}
c_1 &= \int_0^1 \frac{\mathrm{d}x}{\sqrt{6-2x^2}}\left[\ln\frac{3}{4} + \ln\frac{3+x}{2+x} + \frac{x}{2+x}\left(\ln\frac{3+x}{3} + \frac{x}{2-x}\ln\frac{2+x}{4}\right)\right] \\
&= 0.0217376333 \qquad (19)
\end{aligned}
$$

and

$$
c_2 = \frac{\pi^2}{4\sqrt{2}} + \sqrt{2}\int_0^1 \frac{\mathrm{d}x}{\sqrt{1+x^2}}\left[\ln\frac{x}{1+x} + \frac{\ln(1+x)}{x}\right] = 0.973771427\,. \quad (20)
$$

Above T_c, at $M_0^2 = 0$, the coefficients in Eqs. (11)–(16) are taken at $\bar{w} = 1$ and the coefficients F_{310} and F_{320} do not contribute. All other coefficients F_{3lk} vanish. Eqs. (9)–(20) are the main result of our paper. They provide the basis for deriving the analytic form of the amplitude functions of the order parameter and the specific heat for general n. For the case $n = 1$, Eqs. (9)–(20) reduce to the three-loop part of Eq. (3.18) of Bagnuls et al. [7] and of Eq. (3.3) of Halfkann and Dohm [8] and agree with the numerical values of their coefficients F_{blk} for $b = 1$, $b = 2$ and $b = 3$. The latter agree also with those in Eq. (A1.1) of Guida and Zinn-Justin [9] for $n = 1$.

Eq. (9) contains logarithms of the coupling u_0 as expected because of the non-analytic u_0 dependence of δr_0, Eq. (6). Furthermore, the terms proportional

to $n-1$ depend non-analytically on r_{0T} through $\bar{w}^{-1/2}$, $\bar{w}^{1/2}$ and $\bar{w}^{1/2}\ln\bar{w}$ (and higher orders in the neglected terms). These non-analyticities will lead to perturbative terms in the derivatives of the free energy (with respect to M_0) that diverge when the coexistence curve is approached ($T < T_{\mathrm{c}}$, $h_0 \to 0$). This is again the effect of the Goldstone modes that was found previously in two-loop order [6]. For O(n) symmetric quantities, however, these divergences should cancel among themselves [34]. This is indeed the case, at least up to three-loop order, for the square of the order parameter, the Gibbs free energy and for the specific heat as we shall see below.

3 Bare order parameter

The order parameter $M_0(r_0', u_0, h_0)$ is determined by inverting the equation of state,

$$h_0(r_0', u_0, M_0) = \frac{\partial}{\partial M_0}\mathring{\Gamma}(r_0', u_0, M_0)\,. \tag{21}$$

This inversion should be performed iteratively at finite $h_0 \neq 0$. Several perturbative terms of $\partial\mathring{\Gamma}/\partial M_0$ exhibit Goldstone singularities for $h_0 \to 0$ below T_{c} which arise from the three-loop terms $\sim \bar{w}^{-1/2}$, $\bar{w}^{1/2}$ and $\bar{w}^{1/2}\ln\bar{w}$ (after an expansion of $\bar{w}$ dependent terms with respect to u_0 at $h_0 \neq 0$), in addition to the known [6] two-loop singularities, as well as from the expansion of one- and two-loop terms (with respect to u_0 contained in $M_0^2(r_0', u_0, h_0)$ at finite h_0). We have verified that all Goldstone divergences cancel among themselves which leads to a finite result for M_0^2 as a function of r_0' and u_0 in the limit $h_0 \to 0$ at $d = 3$. This function reads for $r_0' < 0$

$$\begin{aligned}
M_0^2 = {} & \frac{1}{8u_0}(-2r_0') + \frac{3}{4\pi}(-2r_0')^{1/2} \\
& + \frac{u_0}{8\pi^2}\left[10 - n + 4(n-1)\ln 3 - 2(n+2)\ln\frac{-2r_0'}{(24u_0)^2}\right] \\
& + \frac{u_0^2(-2r_0')^{-1/2}}{1920\pi^3}\Bigg\{ -2736n - 5904 - 6480c_1 + 240(n-1)c_2 \\
& - (75n^2 - 5n + 875)\pi^2 - 1260\left[(n-1)\,\mathrm{Li}_2\left(\tfrac{1}{3}\right) + 9\,\mathrm{Li}_2\left(-\tfrac{1}{3}\right)\right] \\
& + 960(n-1)\left[2\,\mathrm{Li}_2(-2) - \mathrm{Li}_2\left(-\tfrac{1}{2}\right)\right] - 630(n-1)(\ln 3)^2 \\
& - 48\ln 2\Big[10(n-1)\ln 2 - 60(n-1)\ln 3 + 111n - 561\Big] \\
& + 240(12n-57)\ln 3 - 1440(n+2)\ln\frac{-2r_0'}{(24u_0)^2}\Bigg\}
\end{aligned}$$

$$+ O(u_0^3, u_0^3 \ln u_0)\,. \tag{22}$$

It contains the expected [6] logarithmic u_0 dependence $\sim u_0^2 \ln u_0$ at three-loop order. The logarithmic terms in u_0 can be absorbed by employing the pseudo-correlation length [5,6] ξ_- as a temperature variable below T_c instead of $r_0' < 0$. The relation between r_0' and ξ_- is determined by the two-point vertex function and is given up to three-loop order by (see Appendix B)

$$\begin{aligned} -2r_0' = \xi_-^{-2}\Bigg\{ & 1 + \frac{n+2}{\pi} u_0\xi_- - \frac{n+2}{\pi^2}(u_0\xi_-)^2 \left[\frac{1385}{108} + 4\ln(24u_0\xi_-)\right] \\ & + \frac{n+2}{108\pi^3}(u_0\xi_-)^3 \Bigg[1314n + 13047 + 576(n+8)\,\mathrm{Li}_2\Big(-\tfrac{1}{3}\Big) \\ & + 48(n+8)\pi^2 + 8\,(43n+182)\ln\frac{3}{4}\Bigg] + O(u_0^4\xi_-^4)\Bigg\}\,. \end{aligned} \tag{23}$$

This leads to the bare perturbative expression for the square of the order parameter $M_0^2(\xi_-, u_0, 3)$ that is finite in three dimensions and is free of logarithms of u_0,

$$M_0^2(\xi_-, u_0, 3) = \xi_-^{-1}\left\{\sum_{m=0}^{3} a_{\varphi m}(u_0\xi_-)^{m-1} + O(u_0^3\xi_-^3)\right\} \tag{24}$$

with the coefficients up to two-loop order [6],

$$a_{\varphi 0} = \frac{1}{8}\,, \tag{25}$$

$$a_{\varphi 1} = \frac{1}{8\pi}(n+8)\,, \tag{26}$$

$$a_{\varphi 2} = \frac{1}{2\pi^2}(n-1)\ln 3 - \frac{1}{864\pi^2}(1169n + 1042)\,, \tag{27}$$

and the new three-loop coefficient

$$\begin{aligned} a_{\varphi 3} = \frac{1}{17280\pi^3}\Bigg\{ & 36\,(565n^2 + 5056n + 7744) - 58320c_1 + 2160(n-1)c_2 \\ & + 15\pi^2(19n^2 + 643n + 499) + 180\,(64n^2 + 640n + 457)\,\mathrm{Li}_2\Big(-\tfrac{1}{3}\Big) \\ & - 11340(n-1)\,\mathrm{Li}_2\Big(\tfrac{1}{3}\Big) + 8640(n-1)\left[2\,\mathrm{Li}_2(-2) - \mathrm{Li}_2\Big(-\tfrac{1}{2}\Big)\right] \\ & + 80\,(86n^2 + 860n - 811)\ln 3 - 16\,(860n^2 + 8357n - 7867)\ln 2 \\ & - 5670(n-1)(\ln 3)^2 + 4320(n-1)\Big(6\ln 3 - \ln 2\Big)\ln 2\Bigg\}\,. \end{aligned} \tag{28}$$

For $n = 1$, Eqs. (24)–(28) agree with Eq. (3.13) and the numerical values in Table 2 of Halfkann and Dohm [8].

4 Bare Gibbs free energy

The bare Gibbs free energy $\mathring{\mathcal{F}}$ is determined by the bare Helmholtz free energy and the order parameter as

$$\mathring{\mathcal{F}}(r_0', u_0, h_0) = \mathring{\Gamma}(r_0', u_0, M_0(r_0', u_0, h_0)) - h_0 M_0(r_0', u_0, h_0)\,. \tag{29}$$

Above T_c we obtain the Gibbs free energy $\mathring{\mathcal{F}}_+(r_0', u_0)$ at $h_0 = 0$ by substituting $M_0 = 0$ and $\bar{w} = 1$ into $\mathring{\Gamma}$. From Eqs. (9)–(16) we find at $d = 3$

$$\begin{aligned}\mathring{\mathcal{F}}_+(r_0', u_0) =& \mathring{\Gamma}(r_0', u_0, 0) \\ =& -\frac{n}{12\pi} r_0'^{3/2} - \frac{n(n+2)}{(4\pi)^2} u_0 r_0' - \frac{2n(n+2)}{(4\pi)^3} u_0^2 r_0'^{1/2} \Bigg[n - 6 \\ & - 8 \ln \frac{3}{4} + 4 \ln \frac{r_0'}{(24u_0)^2} \Bigg] + O(u_0^3, u_0^3 \ln u_0)\end{aligned} \tag{30}$$

for $r_0' > 0$. Below T_c we make a perturbative expansion of $\mathring{\mathcal{F}}$, Eq. (29), in u_0 at $h_0 \neq 0$ and take the limit

$$\mathring{\mathcal{F}}_-(r_0', u_0) \equiv \lim_{h_0 \to 0} \mathring{\mathcal{F}}(r_0', u_0, h_0)\,, \quad r_0' < 0\,. \tag{31}$$

The result is at $d = 3$

$$\begin{aligned}\mathring{\mathcal{F}}_-(r_0', u_0) =& -\frac{(-2r_0')^2}{64u_0} - \frac{(-2r_0')^{3/2}}{12\pi} \\ & - \frac{u_0(-2r_0')}{16\pi^2} \left[6 + 2(n-1)\ln 3 - (n+2) \ln \frac{-2r_0'}{(24u_0)^2} \right] \\ & + \frac{u_0^2 (-2r_0')^{1/2}}{384\pi^3} \Bigg[16\,(11n+7) - 1296 c_1 - 48(n-1) c_2 \\ & + (9n^2 + n + 17)\pi^2 + 36(n-1)\,\mathrm{Li}_2\Big(\tfrac{1}{3}\Big) + 324\,\mathrm{Li}_2\Big(-\tfrac{1}{3}\Big) \\ & + 96(n-1)\left[\mathrm{Li}_2\Big(-\tfrac{1}{2}\Big) - 2\,\mathrm{Li}_2(-2) \right] + 18(n-1)(\ln 3)^2 \\ & + 48(n-1)\Big(\ln 2 - 6 \ln 3 \Big) \ln 2 - 48\,(4n+17) \ln 3 \\ & + 16\,(31n+95) \ln 2 + 96(n+2) \ln \frac{-2r_0'}{(24u_0)^2} \Bigg]\end{aligned}$$

$$+O(u_0^3, u_0^3 \ln u_0)\,. \tag{32}$$

As expected, all Goldstone divergences cancel among themselves in this limit. The terms of $O(\bar{w}^{1/2})$, $O(\bar{w}^{1/2}\ln\bar{w})$ and $O(\bar{w})$ in Eqs. (15)–(18) do not contribute to the free energy $\mathring{\mathcal{F}}_-(r_0', u_0)$ and to the specific heat $\partial^2\mathring{\mathcal{F}}_-(r_0', u_0)/(\partial r_0')^2$ on the coexistence curve because these terms vanish in the limit $h_0 \to 0$ ($\bar{w} \to 0$). Therefore, for the purpose of an application to the specific heat in Ref. [10], it was sufficient to evaluate most of the diagrams in Fig. 1 directly at $r_{0T} = 0$ ($\bar{w} = 0$), see also App. A. This simplification, however, is not applicable to the calculation of the order parameter performed in the present paper.

We note that the function $\mathring{\mathcal{F}}_-(r_0', u_0)$ can also be considered as the Helmholtz free energy $\mathring{\Gamma}(r_0', u_0, M_0)$ below T_c in the limit where M_0 approaches the spontaneous value $M_0(r_0', u_0)$, Eq. (22), of the order parameter on the coexistence curve. Thus the Helmholtz free energy remains finite in this limit, as expected [34]. In order to express $\mathring{\mathcal{F}}_\pm$ in terms of the correlation lengths $\xi_\pm$ we use (see Appendix B)

$$\begin{aligned} r_0' = \xi_+^{-2}\Bigg\{ & 1 + \frac{n+2}{\pi}u_0\xi_+ + \frac{n+2}{\pi^2}(u_0\xi_+)^2\left[\frac{1}{27} + 2\ln(24u_0\xi_+)\right] \\ & + \frac{n+2}{54\pi^3}(u_0\xi_+)^3\Bigg[3\,(3n+22) - 144(n+8)\,\mathrm{Li}_2\Big(-\tfrac{1}{3}\Big) \\ & - 12(n+8)\pi^2 - 2\,(43n+182)\ln\frac{3}{4}\Bigg] + O(u_0^4\xi_+^4)\Bigg\} \end{aligned} \tag{33}$$

for $r_0' > 0$ and $r_0'(\xi_-, u_0)$ as given in Eq. (23) for $r_0' < 0$. (We recall that, for $d = 3$, the right-hand side of Eq. (33) contains a logarithmic u_0 dependence only at $O(u_0^2)$, as can be seen in Eq. (4.5) of Ref. [20] for $\epsilon = 1$. The same comment applies to Eq. (23).) Substituting Eqs. (33) and (22) into Eqs. (30) and (32) yields the Gibbs free energy up to three-loop order

$$\begin{aligned} \mathring{\mathcal{F}}_+(\xi_+, u_0) = \xi_+^{-3}\Bigg\{ & -\frac{n}{12\pi} - \frac{3n(n+2)}{16\pi^2}\,(u_0\xi_+) - \frac{n(n+2)}{8\pi^3}\left[\frac{1}{27} + n\right. \\ & \left. + \ln\frac{16}{9}\right](u_0\xi_+)^2 + O\Big(u_0^3\xi_+^3, u_0^3\xi_+^3\ln(u_0\xi_+)\Big)\Bigg\} \end{aligned} \tag{34}$$

and

$$\mathring{\mathcal{F}}_-(\xi_-, u_0) = \xi_-^{-3}\left\{\sum_{m=0}^{3} a_{-m}^{(\Gamma)}(u_0\xi_-)^{m-1} + O\Big(u_0^3\xi_-^3, u_0^3\xi_-^3\ln(u_0\xi_-)\Big)\right\} \tag{35}$$

with the coefficients up to two-loop order

$$a^{(\Gamma)}_{-0} = -\frac{1}{64}\,, \tag{36}$$
$$a^{(\Gamma)}_{-1} = -\frac{1}{96\pi}\,(3n+14)\,, \tag{37}$$
$$a^{(\Gamma)}_{-2} = -\frac{1}{3456\pi^2}\left[54n^2 - 737n - 394 + 432(n-1)\ln 3\right] \tag{38}$$

and the new three-loop coefficient

$$\begin{aligned} a^{(\Gamma)}_{-3} = \frac{1}{3456\pi^3}\Big[& 179n^2 - 3875n - 10626 + (33n^2 - 471n - 615)\pi^2 \\ & - 11664c_1 - 432(n-1)c_2 - 12\,(48n^2 + 480n + 525)\,\mathrm{Li}_2\left(-\tfrac{1}{3}\right) \\ & + 540(n-1)\,\mathrm{Li}_2\left(-\tfrac{1}{2}\right) - 1728(n-1)\,\mathrm{Li}_2(-2) \\ & + 54(n-1)\left[5\ln 2 - 42\ln 3\right]\ln 2 + 16\,(43n^2 + 547n + 1219)\ln 2 \\ & - 8\,(97n^2 + 538 + 1174)\ln 3\Big]\,. \end{aligned} \tag{39}$$

For $n=1$ this Gibbs free energy is given for $T < T_c$ in Eqs. (3.14), (3.15) and Table 2 of Ref. [8] where it is denoted by $\tilde{\Gamma}_{-0}(\xi_-, u_0)$. The logarithmic contributions $O(u_0^3 \ln(u_0\xi_\pm))$ in Eqs. (34) and (35) correspond to Eq. (3.15) of Ref. [8] and are due to a specific $d=3$ divergent diagram arising only at four-loop order [7]. Its pole term is to be subtracted within the overall subtractions mentioned in Section 2.

5 Bare specific heat

The specific heat abve (+) and below (−) T_c at $h_0 = 0$ is determined by [5,6]

$$\mathring{C}^{\pm} = C_B - a_0^2\,\mathring{\Gamma}^{(2,0)}_{\pm}(r_0', u_0) \tag{40}$$

where

$$\mathring{\Gamma}^{(2,0)}_{\pm}(r_0', u_0) = \frac{\partial^2}{(\partial r_0')^2}\,\mathring{\mathcal{F}}_{\pm}(r_0', u_0)\,. \tag{41}$$

C_B is a noncritical background value and a_0 is a constant defined by $r_0 - r_{0c} = a_0 t$ where $t = (T - T_c)/T_c$. The result is

$$\mathring{\Gamma}^{(2,0)}_{+}(r_0', u_0) = -\frac{n}{16\pi} r_0'^{\,-1/2} + \frac{n(n+2)}{2(4\pi)^3} u_0^2 r_0'^{\,-3/2}\Big[n - 6 - 8\ln\frac{3}{4}$$

$$+ 4 \ln \frac{r'_0}{(24u_0)^2}\Bigg] + O(u_0^3, u_0^3 \ln u_0) \tag{42}$$

and

$$\begin{aligned}
\mathring{\Gamma}_-^{(2,0)}(r'_0, u_0) = & -\frac{1}{8u_0} - \frac{1}{4\pi}(-2r'_0)^{-1/2} + \frac{1}{4\pi^2}(n+2)u_0(-2r'_0)^{-1} \\
& - \frac{u_0^2(-2r'_0)^{-3/2}}{384\pi^3}\Bigg[16\,(11n+7) - 1296c_1 - 48(n-1)c_2 \\
& + (9n^2+n+17)\pi^2 + 36(n-1)\,\mathrm{Li}_2\Big(\tfrac{1}{3}\Big) + 324\,\mathrm{Li}_2\Big(-\tfrac{1}{3}\Big) \\
& + 96(n-1)\,\Big[\mathrm{Li}_2\Big(-\tfrac{1}{2}\Big) - 2\,\mathrm{Li}_2(-2)\Big] + 18(n-1)(\ln 3)^2 \\
& + 48(n-1)\Big(\ln 2 - 6\ln 3\Big)\ln 2 - 48\,(4n+17)\ln 3 \\
& + 16\,(31n+95)\ln 2 + 96(n+2)\ln \frac{-2r'_0}{(24u_0)^2}\Bigg] \\
& + O(u_0^3, u_0^3 \ln u_0)\,.
\end{aligned} \tag{43}$$

Unlike the corresponding two-loop result [6] for $\mathring{\Gamma}_\pm^{(2,0)}$, the three-loop terms of $\mathring{\Gamma}_\pm^{(2,0)}$ contain a logarithmic u_0-dependence (as long as they are considered as a function of r'_0). This can be absorbed by turning to the formulation in terms of the correlation lengths $\xi_\pm$ [5,20]. The relation between r'_0 and $\xi_\pm$ is given in Eqs. (33) and (23). This leads to the bare perturbative expression for the specific heat at $h_0 = 0$ that is finite in three dimensions and is free of logarithms in u_0,

$$\begin{aligned}
\mathring{\Gamma}_+^{(2,0)}(\xi_+, u_0, 3) = \xi_+\Bigg\{ & -\frac{n}{16\pi} + \frac{n(n+2)}{32\pi^2}u_0\xi_+ \\
& - \frac{n(n+2)}{1728\pi^3}(u_0\xi_+)^2\left[27n + 160 + 108\ln\frac{3}{4}\right] + O(u_0^3\xi_+^3)\Bigg\}
\end{aligned} \tag{44}$$

and

$$\mathring{\Gamma}_-^{(2,0)}(\xi_-, u_0, 3) = \xi_-\left\{\sum_{m=0}^{3} a_{-m}^{(2,0)}(u_0\xi_-)^{m-1} + O(u_0^3\xi_-^3)\right\} \tag{45}$$

with the coefficients up to two-loop order [6],

$$a_{-0}^{(2,0)} = -\frac{1}{8}\,, \tag{46}$$

$$a_{-1}^{(2,0)} = -\frac{1}{4\pi}\,, \tag{47}$$

$$a_{-2}^{(2,0)} = \frac{3}{8\pi^2}(n+2)\,, \tag{48}$$

and the new three-loop coefficient

$$\begin{aligned} a_{-3}^{(2,0)} = -\frac{1}{3456\pi^3}\Big[& 1188n^2 + 11876n + 16840 - 11664c_1 - 432(n-1)c_2 \\ & + 9\pi^2(9n^2+n+17) + 324(n-1)\,\mathrm{Li}_2\Big(\tfrac{1}{3}\Big) + 2916\,\mathrm{Li}_2\Big(-\tfrac{1}{3}\Big) \\ & + 864(n-1)\Big[\mathrm{Li}_2\Big(-\tfrac{1}{2}\Big) - 2\,\mathrm{Li}_2(-2)\Big] + 162(n-1)(\ln 3)^2 \\ & + 432(n-1)\Big(\ln 2 - 6\ln 3\Big)\ln 2 - 432\,(4n+17)\ln 3 \\ & + 144\,(31n+95)\ln 2\Big]\,. \end{aligned} \tag{49}$$

For the case $n = 1$, Eqs. (45)–(49) agree with Eq. (3.17) and the numerical values in Table 2 of Halfkann and Dohm [8].

6 Renormalization and amplitude functions in three dimensions

In the preceding Sections no renormalizations have been used, except for the shift of the temperature variable r_0 by δr_0, Eq. (6), and for the additive subtraction of the Helmholtz free energy. All other quantities are unrenormalized. Because of the super-renormalizability of the ϕ^4 theory for $d < 4$, the shift of δr_0 and the additive subtraction are sufficient to make the order parameter $M_0^2(\xi_-, u_0, 3)$ and the vertex function $\mathring{\Gamma}_{\pm}^{(2,0)}(\xi_\pm, u_0, 3)$ finite at infinite cutoff in three dimensions as long as $\xi_\pm$ is finite. The resulting bare perturbation series in Eqs. (24), (44) and (45), however, are obviously not applicable near T_c where $u_0\xi_\pm$ diverges, thus the series have to be mapped from the critical to the non-critical region.

This mapping is achieved by turning to the renormalized theory as defined below and by introducing the renormalization scale μ which can be varied via the renormalization-group equation (RGE) [35]. The solution of the RGE implies a decomposition of thermodynamic quantities into a product of amplitude functions and exponential parts. This decomposition is most natural and simple within the minimal subtraction scheme [36] where the exponential parts are determined entirely from the pure $d = 4$ pole terms $\sim \epsilon^{-n}$ of the renormalization constants $Z_r(u,\epsilon)$, $Z_u(u,\epsilon)$, $Z_\phi(u,\epsilon)$ and $A(u,\epsilon)$ [5,20]. As shown previously [5,17,20], the use of these pure pole terms does not imply the necessity of working near four dimensions and of using the ϵ-expansion. This is a nontrivial aspect of the minimal subtraction scheme that is not widely appreciated in the field-theoretic literature. Here we make use of the fact that

the minimal subtraction scheme can well be combined with the concept of a ϕ^4 theory at fixed dimension $d < 4$.

Our $d = 3$ approach differs from the $d = 3$ approach of previous work [7,9,21–24] where renormalization conditions were used. These renormalization conditions are different above and below T_c whereas the minimal renormalization applies to both $T > T_\mathrm{c}$ and $T < T_\mathrm{c}$ in the same form. Within our scheme the minimally renormalized quantities are introduced in d dimensions as

$$r = Z_r^{-1}(r_0 - r_{0c}) \, , \quad u = \mu^{-\epsilon} A_d Z_u^{-1} Z_\phi^2 u_0 \, , \quad \vec{\phi} = Z_\phi^{-1/2} \vec{\phi}_0 \, , \tag{50}$$

$$M^2(\xi_-, u, \mu, d) = Z_\phi^{-1} M_0^2(\xi_-, \mu^\epsilon A_d^{-1} Z_u Z_\phi^{-2} u, d) \, , \tag{51}$$

$$\Gamma_\pm^{(2,0)}(\xi_\pm, u, \mu, d) = Z_r^2 \, \mathring{\Gamma}_\pm^{(2,0)}(\xi_\pm, \mu^\epsilon Z_u Z_\phi^{-2} A_d^{-1} u, d) \, - \, \frac{1}{4} \mu^{-\epsilon} A_d A(u, \epsilon), \tag{52}$$

where $\xi_\pm$ are the correlation lengths defined in Refs. [5,20] and

$$A_d = \frac{\Gamma(3 - d/2)}{2^{d-2} \pi^{d/2} (d-2)} \tag{53}$$

is a convenient geometric factor [17]. We recall that the amplitude functions to be defined in Eqs. (54) and (55) depend on the choice of this geometric factor. In our calculations, Eqs. (50)–(53) are used directly at $d = 3$. The analytic form of the renormalization constants $Z_r(u, \epsilon)$, $Z_u(u, \epsilon)$, $Z_\phi(u, \epsilon)$ and $A(u, \epsilon)$ is given in Eqs. (2.13), (2.16)–(2.19) and (B1)–(B18) of Ref. [10] for general n up to five-loop order. As is well known, the minimal renormalization constants in Eqs. (50)–(52) have the property of absorbing the remaining ultraviolet divergences for $d \to 4$ that were not absorbed by the mass shift and subtractions mentioned in Sect. 2. In the context of our $d = 3$ theory this property is, of course, irrelevant. The crucial aspect of the renormalization of Eqs. (50)–(52) at $\epsilon = 1$ is that they provide the mapping and decomposition mentioned above via the integration of the RGE for the renormalized quantities M^2 and $\Gamma_\pm^{(2,0)}$. The critical behavior of these quantities then evolves from the *infrared* divergences of the Z-factors $Z_i(u, 1)$ as $u \to u^\star$ [20,23] and from the singular behavior of $A(u, 1)$ as $u \to u^\star$ [26].

Dimensionless amplitude functions f_ϕ, F_+ and F_- of the renormalized order parameter and specific heat can be defined at fixed dimension $2 < d < 4$ according to

$$f_\phi(\mu\xi_-, u, d) = \xi_-^{d-2} M^2(\xi_-, u, \mu, d) \tag{54}$$

and

$$F_\pm(\mu\xi_\pm, u, d) = -4\mu^\epsilon A_d^{-1} \Gamma_\pm^{(2,0)}(\xi_\pm, u, \mu, d) \, . \tag{55}$$

These functions remain finite also in the limit $d \to 4$ (at finite $\xi_\pm$) [5,20] provided that the appropriate subtractions (including all additive pole terms for $d < 4$) have been performed in Sect. 2. In the application of the solution of the RGE the amplitude functions appear in the form [5] with the non-critical arguments $\mu\xi_\pm = 1$ in three dimensions

$$f_\phi(u) \equiv f_\phi(1, u, 3)\,, \quad F_\pm(u) \equiv F_\pm(1, u, 3)\,. \tag{56}$$

These functions should be smooth and well behaved in the entire region $0 \leq u \leq u^\star$ [20]. They have the power series

$$f_\phi(u) = \frac{1}{u} \sum_{m=0}^{\infty} c^-_{\varphi m} u^m\,, \tag{57}$$

$$F_+(u) = \sum_{m=0}^{\infty} c^+_{Fm} u^m\,, \tag{58}$$

$$F_-(u) = \frac{1}{u} \sum_{m=0}^{\infty} c^-_{Fm} u^m \tag{59}$$

with n-dependent coefficients. These series have a zero radius of convergence but are (presumably) Borel resummable [20,28] (see Sect. 7). Note that the coefficients of these series depend on the choice of the geometric factor A_d^{-1}. Our choice, Eq. (53), minimizes the explicit dimensional dependence of the lowest order coefficients in Eqs. (57)–(59) [5,17,20,37] and is expected to improve the convergence properties of the series. This is of relevance in the context of Borel resummations based on low-order information.

From Eqs. (24), (44) and (45) we obtain the analytic expressions of the coefficients for the order parameter

$$c^-_{\varphi 0} = \frac{1}{32\pi}\,, \tag{60}$$

$$c^-_{\varphi 1} = 0\,, \tag{61}$$

$$c^-_{\varphi 2} = \frac{1}{27\pi}(160 - 82n) + \frac{2}{\pi}(n-1)\ln 3\,, \tag{62}$$

$$\begin{aligned} c^-_{\varphi 3} = \frac{-1}{1080\pi}\Big\{ & 2500n^2 + 65104n + 29056 + 8640\,(5n+22)\zeta(3) + 58320c_1 \\ & + 2160(n-1)\left[4\,\mathrm{Li}_2\!\left(-\tfrac{1}{2}\right) - c_2\right] - 15\pi^2(19n^2 + 643n + 499) \\ & - 180\,(64n^2 + 640n + 457)\,\mathrm{Li}_2\!\left(-\tfrac{1}{3}\right) + 11340(n-1)\,\mathrm{Li}_2\!\left(\tfrac{1}{3}\right) \\ & - 17280(n-1)\,\mathrm{Li}_2(-2) + 5670(n-1)(\ln 3)^2 + 4320(n-1)(\ln 2)^2 \\ & - 25920(n-1)(\ln 2)(\ln 3) - 80\,(194n^2 + 1616n - 1675)\ln 3 \end{aligned}$$

$$+ 16\,(860n^2 + 8357n - 7867)\ln 2 \Bigg\} , \tag{63}$$

for the specific heat above T_c,

$$c^+_{F0} = -n\,, \tag{64}$$
$$c^+_{F1} = -2n(n+2)\,, \tag{65}$$
$$c^+_{F2} = -4n(n+2)\left[n - \frac{7}{27} + 4\ln\frac{4}{3}\right] , \tag{66}$$

and for the specific heat below T_c,

$$c^-_{F0} = \frac{1}{2}\,, \tag{67}$$
$$c^-_{F1} = -4\,, \tag{68}$$
$$c^-_{F2} = 8(10-n)\,, \tag{69}$$
$$\begin{aligned} c^-_{F3} = & -\frac{1}{27}(1080n^2 + 3464n + 31120) - 128\,(5n+22)\zeta(3) - 864c_1 \\ & - 32(n-1)c_2 + \frac{\pi^2}{3}(18n^2 + 2n + 34) + 24(n-1)\,\mathrm{Li}_2\!\left(\tfrac{1}{3}\right) \\ & + 216\,\mathrm{Li}_2\!\left(-\tfrac{1}{3}\right) + 64(n-1)\,\mathrm{Li}_2\!\left(-\tfrac{1}{2}\right) - 128(n-1)\,\mathrm{Li}_2(-2) \\ & + 12(n-1)(\ln 3)^2 + 32(n-1)(\ln 2)^2 - 192(n-1)(\ln 2)(\ln 3) \\ & - 32\,(4n+17)\ln 3 + \frac{32}{3}(31n+95)\ln 2\,. \end{aligned} \tag{70}$$

The terms up to $O(u)$ are the previous two-loop results [5,6]. In Table 1 we give the numerical values of the coefficients $c^-_{\varphi m}$, c^+_{Fm} and c^-_{Fm} up to three-loop order for $n = 1, 2, 3$ as obtained from Eqs. (60)–(70).

For $n = 1$ the numerical values of the coefficients $c^-_{\varphi m}$ given previously [8] agree with ours up to eight digits. The difference in the ninth digit is due to a numerical inaccuracy of the Z-factors in three dimensions that were available previously only in numerical form [38]. (A similar comments applies to the numerical values of the series for $n = 1$ [8] mentioned in the preceding Sections.) In the present work we have used the Z-factors in the analytic form for general n as given in Ref. [10].

Numerical results of a six-loop calculation (above T_c for $n = 1, 2, 3$) and of a five-loop calculation (below T_c for $n = 1$) of the diagrams contributing to $F_\pm$ in three dimensions have been presented by Baker et al. [22], by Bagnuls and Bervillier [23] and by Bagnuls et al. [7], respectively. On this basis the coefficients c^+_{Fm} for $n = 1, 2, 3$ and c^-_{Fm} for $n = 1$ were calculated in numerical form by Krause et al. [38] and by Halfkann and Dohm [8] who used a two-loop approximation for the additive renormalization constant $A(u,\epsilon)$. The

Table 1
Coefficients $c^{-}_{\varphi m}$ of $f_\phi(u)$ according to Eqs. (57), (60)–(63), and $c^{\pm}_{Fm}$ of $F_+(u)$ and $F_-(u)$ according to Eqs. (58), (64)–(66) and (59), (67)–(70), respectively, for $n = 1$, 2, 3 up to three-loop order. For $c^{-}_{\varphi m}$ and c^{-}_{Fm}, m refers to u^{m-1} corresponding to m-loop order whereas for c^{+}_{Fm}, m refers to u^m corresponding to $(m+1)$-loop order.

	m	$c^{-}_{\varphi m}$	c^{+}_{Fm}	c^{-}_{Fm}
$n = 1$	0	$(32\pi)^{-1}$	-1	0.5
	1	0	-6	-4
	2	0.919561893	-22.6976284	72
	3	-80.8015258		-5189.75474
$n = 2$	0	$(32\pi)^{-1}$	-2	0.5
	1	0	-16	-4
	2	0.652241285	-92.5270090	64
	3	-83.0064428		-5918.07320
$n = 3$	0	$(32\pi)^{-1}$	-3	0.5
	1	0	-30	-4
	2	0.384920676	-233.488142	56
	3	-82.6969869		-6607.95641

corrected coefficients for c^{+}_{Fm} for $n = 1$, 2, 3 and c^{-}_{Fm} for $n = 1$ using $A(u, \epsilon)$ in five-loop order have been presented recently in numerical form [10]. Our present results, Eqs. (64)–(70), provide the analytic form of these coefficients up to three-loop order for general n. The three-loop coefficient c^{+}_{F2} has been calculated independently by Burnett [39].

7 Results and discussion

Within the minimally renormalized ϕ^4 field theory in three dimensions we have derived the three-loop contributions to the amplitude functions $f_\phi(u)$ of the square of the order parameter, Eqs. (57), (63), and to the amplitude functions $F_+(u)$ and $F_-(u)$ of the specific heat above and below T_c, Eqs. (58), (66) and (59), (70) for general n. Similar to the previous results below T_c for $n = 1$ [7,8,10], the coefficients $c^{-}_{\varphi m}$ and c^{-}_{Fm} for $n = 2$ and $n = 3$ given in Table 1 have alternating signs and increase considerably in magnitude. Clearly a resummation of these series is necessary in order to obtain quantitatively

reliable results. For a description of the method of Borel resummation we refer to Refs. [20,28] and, in the present context, to Ref. [10].

While the previous two-loop results [6] did not yet provide sufficient information for a controlled resummation procedure and thus did not yet lead to an error estimate, the new information on $c^-_{\varphi 3}$ and c^-_{F3} presented here makes possible to perform the Borel resummation for $n > 1$ below T_c with reasonably small error bars. The reliability of this Borel resummation based on three-loop results (with four low-order coefficients, see Table 1) has been demonstrated in detail in a related paper [10]. We refer to this paper for a description of the method of determining the error bars. For the case $n = 2$ (corresponding to the superfluid transition of ^{4}He) the resummation result is

$$u^\star F_-(u^\star) = 0.384 \pm 0.025 \tag{71}$$

at the fixed point $u^\star = 0.0362$ [10]. In addition, using the new coefficient $c^-_{\varphi 3}$ for $n = 2$ and $n = 3$, a Borel resummation of the fixed point value of the amplitude function $f_\phi(u^\star)$ has been performed by Mönnigmann [40]. (For this case the Borel resummation parameters b and α [20,28] are $b = 3.5 + n/2 = 4.5$ and $-0.43 \leq \alpha \leq 0.47$ for $n = 2$, and $b = 5$, $-0.62 \leq \alpha \leq 0.71$ for $n = 3$.) The result is

$$u^\star f_\phi(u^\star) = 0.010099 \pm 0.000084\,, \quad n = 2\,, \tag{72}$$
$$u^\star f_\phi(u^\star) = 0.00997 \pm 0.00011\,, \quad n = 3 \tag{73}$$

at the fixed point [40]. Thus the deviation from the zero-loop term $c^-_{\varphi 0} = (32\pi)^{-1} \approx 0.009947$ is very small, confirming the previous expectation [5].

The amplitude functions are of physical relevance not only at the fixed point but also in a finite interval $0 < u < u^\star$ corresponding to a finite distance from criticality where non-asymptotic corrections become non-negligible [18]. Since the Borel resummed amplitude functions $F_-(u)$ and $f_\phi(u)$ are smooth and slowly varying functions for $n = 1$ (see Figs. 1 and 4 of Ref. [8]) the same property should hold also for $n = 2$ and 3. Empirical evidence in support of this fact comes from analyses of experimental data of ^{4}He in the non-asymptotic region [16,18,41]. Therefore, within the accuracy of the present results, these functions can be approximated by the simple representations [8,38]

$$F_-(u) = (2u)^{-1} - 4(1 + d_F u) \tag{74}$$

and

$$f_\phi(u) = A_3 (8u)^{-1}(1 + d_\varphi u)\,, \tag{75}$$

with $A_3 = (4\pi)^{-1}$. Here the lowest-order terms are chosen to coincide with the one-loop results. The coefficients d_F and d_φ are determined by requiring that Eqs. (74) and (75) agree with the Borel-resummed fixed point values of Eqs. (71) and (72). This implies

$$d_F = -5.49\,, \quad d_\varphi = 0.422 \tag{76}$$

for $n = 2$. In view of the present error bars, a more refined representation of $F_-(u)$ and $f_\phi(u)$ does not seem to be warranted until four-loop (Borel-resummed) results with smaller error bars become available.

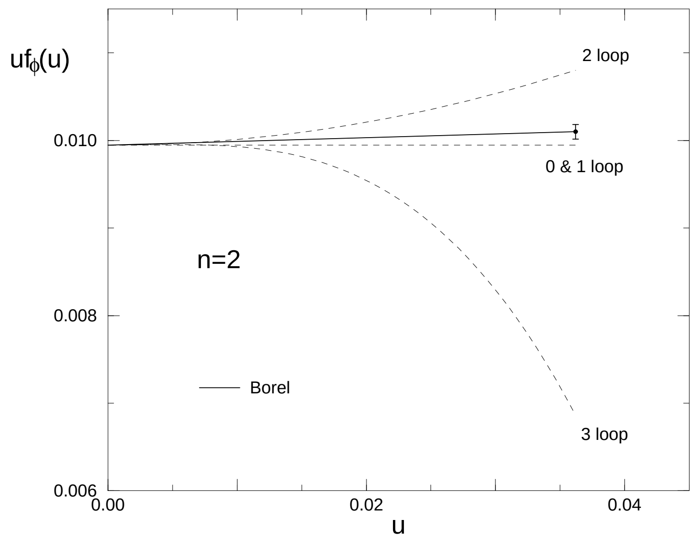

Fig. 2. Partial sums $f_\phi^{(M)}(u) = \frac{1}{u}\sum_{m=0}^{M} c_{\varphi m}^- u^m$ of the amplitude function $f_\phi(u) \equiv f_\phi(1, u, 3)$, Eq. (57), for the square of the order parameter in three dimensions for $n = 2$ multiplied by u, as a function of the renormalized coupling u from $M = 0$ (zero-loop order) to $M = 3$ (three-loop order) (dashed lines). The solid curve is Eq. (75) with $d_\varphi = 0.422$ representing the Borel summation result. The error bar indicates the error of the Borel resummed fixed point value $f_\phi(u^\star)$ at $u^\star = 0.0362$, Eq. (72).

In Figs. 2 and 3 we have plotted $f_\phi(u)$ and $F_-(u)$ for $n = 2$ according to the representations in Eqs. (74) and (75) (solid lines), together with the zero-, one-, two- and three-loop approximations in the range $0 \le u \le u^\star$ (dashed lines). The amplitude functions $f_\phi(u)$ and $F_\pm(u)$ enable us to calculate and predict (rather than fit) the various ratios of amplitudes of leading and subleading terms of the critical temperature dependence of the order parameter and the specific heat [5]. These ratios are universal quantities [13].

As a first application we consider the specific heat which plays an important role in testing the renormalization-group prediction of critical-point universality along the λ-line of ^{4}He [11,12]. Combining the three-loop result for $F_-(u^\star)$, Eq. (71), with the recent [10] five-loop result for the additive renormalization

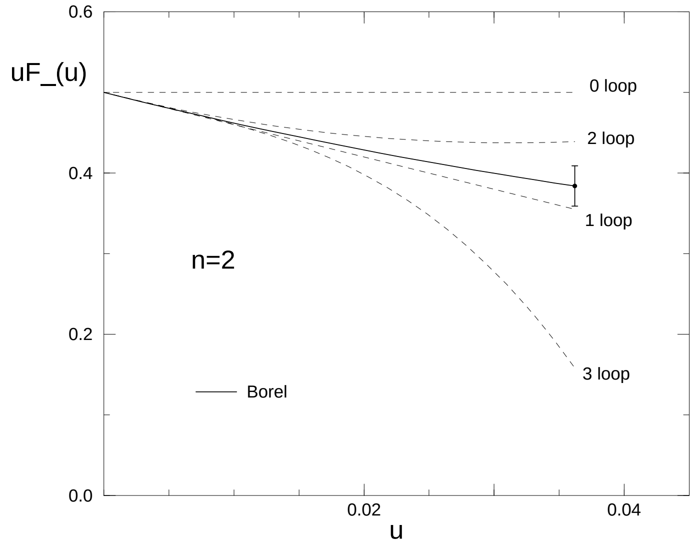

Fig. 3. Partial sums $F_-^{(M)}(u) = \frac{1}{u}\sum_{m=0}^{M} c_{Fm}^- u^m$ of the amplitude function $F_-(u) \equiv F_-(1, u, 3)$, Eq. (59), for the specific heat below T_c in three dimensions for $n = 2$ multiplied by u, as a function of the renormalized coupling u from $M = 0$ (zero-loop order) to $M = 3$ (three-loop order) (dashed lines). The solid curve is Eq. (74) with $d_F = -5.49$ representing the Borel summation result. The error bar indicates the error of the Borel resummed fixed point value $F_-(u^\star)$ at $u^\star = 0.0362$, Eq. (71).

constant $A(u, \epsilon)$ leads to the universal amplitude ratio [10]

$$\frac{A^+}{A^-} = 1.056 \pm 0.004 \tag{77}$$

for the asymptotic specific heat in three dimensions. This is in excellent agreement with the high-precision experimental result $A^+/A^- = 1.054 \pm 0.001$ for ^{4}He near the superfluid transition obtained from a recent experiment in space [15]. Our result, Eq. (77), is a significant improvement over the previous prediction [4] $A^+/A^- = 1.0294 \pm 0.0134$ obtained from the $\epsilon = 4 - d$ expansion up to $O(\epsilon^2)$ [42].

As a second application we employ our Borel-resummed three-loop result for f_ϕ to calculate the universal combination of amplitudes [13]

$$R_C \equiv \frac{A_\chi A^+}{A_M^2} = \frac{(2\nu P_+^\star)^{2-2\beta}}{(3/2 - 2\nu P_+^\star)^{2\beta}} \cdot \frac{4\nu B^\star + \alpha F_+^\star}{16\pi} \left[f_\phi^\star f^{(2)\star} \right]^{-1} \tag{78}$$

where A_χ and A_M are the leading amplitudes of the susceptibility above T_c and of the order parameter below T_c, respectively [13]. The structure of the right-handed side of Eq. (78) follows from previous work [5,38]. Here $f_\phi^\star = f_\phi(u^\star)$ is determined by Eqs. (72) and (73). The quantities $P^\star = P(u^\star)$, $F_+^\star = F_+(u^\star)$,

$B^\star = B(u^\star)$ and $f^{(2)\star} = f^{(2)}(u^\star)$ have been calculated in Refs. [10] and [38], respectively. The values for the critical exponents α and ν are given in Ref. [10], the value of β is taken from Ref. [43]. The resulting values for R_C in three dimensions are

$$R_C = 0.123 \pm 0.003\,, \quad n = 2\,, \tag{79}$$
$$R_C = 0.189 \pm 0.009\,, \quad n = 3\,. \tag{80}$$

These results are significantly more accurate than the ϵ-expansion values [29] up to $O(\epsilon^2)$ whose extrapolation to $\epsilon = 1$ suffers from considerable ambiguities [29]. The numerical estimate $R_C = 0.165$ of high-temperature series expansions for $n = 3$ [44] is not far from our value, Eq. (80). Additional numerical and experimental studies would be desirable.

Further applications of our results will be given elsewhere.

Acknowledgements

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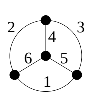
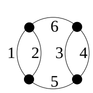
A Three-loop vacuum diagrams

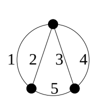
In this Appendix we present some details of the calculation (at infinite cutoff) of the three-loop vacuum diagrams shown in Fig. 1 in three dimensions. There are six different topologies of vacuum diagrams (denoted as (A), (B), …(F) in Ref. [27]) whose integral expressions are given by



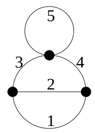
$$= \int_{\mathbf{p}_1}\int_{\mathbf{p}_2}\int_{\mathbf{p}_3} \frac{1}{(m_1^2+p_1^2)(m_2^2+|\mathbf{p}_1-\mathbf{p}_2|^2)(m_3^2+p_2^2)(m_4^2+|\mathbf{p}_2-\mathbf{p}_3|^2)(m_5^2+p_3^2)(m_6^2+|\mathbf{p}_3-\mathbf{p}_1|^2)}\,, \tag{A.1}$$



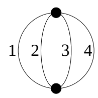
$$= \int_{\mathbf{p}_1}\int_{\mathbf{p}_2}\int_{\mathbf{p}_3} \frac{1}{(m_1^2+p_1^2)(m_2^2+|\mathbf{p}_3-\mathbf{p}_1|^2)(m_3^2+|\mathbf{p}_3-\mathbf{p}_2|^2)(m_4^2+p_2^2)(m_5^2+p_3^2)(m_6^2+p_3^2)}\,, \tag{A.2}$$



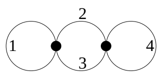
$$= \int_{\mathbf{p}_1}\int_{\mathbf{p}_2}\int_{\mathbf{p}_3} \frac{1}{(m_1^2+p_1^2)(m_2^2+|\mathbf{p}_3-\mathbf{p}_1|^2)(m_3^2+|\mathbf{p}_3-\mathbf{p}_2|^2)(m_4^2+p_2^2)(m_5^2+p_3^2)}\,, \tag{A.3}$$



$$= \int\!\!\!\int\!\!\!\int_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3} \frac{1}{(m_1^2+p_1^2)(m_2^2+p_2^2)(m_3^2+|\mathbf{p}_1+\mathbf{p}_2|^2)(m_4^2+|\mathbf{p}_1+\mathbf{p}_2|^2)(m_5^2+p_3^2)}\,, \tag{A.4}$$



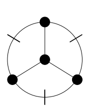
$$= \int\!\!\!\int\!\!\!\int_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3} \frac{1}{(m_1^2+p_1^2)(m_2^2+p_2^2)(m_3^2+p_3^2)(m_4^2+|\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3|^2)}\,, \tag{A.5}$$



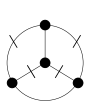
$$= \int\!\!\!\int\!\!\!\int_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3} \frac{1}{(m_1^2+p_1^2)(m_2^2+p_2^2)(m_3^2+p_2^2)(m_4^2+p_3^2)}\,. \tag{A.6}$$

In our case, the masses m_i^2 are given by r_{0L} or r_{0T} of Eq. (8). Three of the diagrams in Fig. 1 are infrared divergent in the limit $r_{0T} \to 0$ in three dimensions: type (B) with $m_1^2 = m_4^2 = r_{0L}$ and $m_2^2 = m_3^2 = m_5^2 = m_6^2 = r_{0T}$, type (D) with $m_1^2 = m_2^2 = m_5^2 = r_{0L}$ and $m_3^2 = m_4^2 = r_{0T}$ and type (F) with $m_1^2 = m_4^2 = r_{0L}$ and $m_2^2 = m_3^2 = r_{0T}$. The leading r_{0T} dependence of these diagrams appears in the terms of $O(\bar{w}^{-1/2})$ in Eqs. (15), (17) and (18). All other diagrams in Fig. 1 are free of infrared divergences in the limit $r_{0T} \to 0$ at $d = 3$.

Most difficult is the evaluation of the "Mercedes" diagram (A), Eq. (A.1). It has been calculated by Rajantie [27] in three dimensions only for two special cases (i) where one of the masses m_i^2 vanishes, or (ii) where all the masses are equal. The case (ii) corresponds to our first Mercedes diagram in Fig. 1 with $m_i^2 = r_{0L}$ and is given by Eq. (55) of Ref. [27]. The last two Mercedes diagrams in Fig. 1, however, contain the two different (longitudinal and transverse) masses r_{0L} and r_{0T}. This case is considerably more difficult than the cases (i) and (ii) mentioned above. We have been able to calculate these diagrams for small $\bar{w} = r_{0T}/r_{0L} > 0$ in three dimensions as



$$\begin{aligned} = \frac{r_{0L}^{-3/2}}{(4\pi)^3} \Bigg[& \frac{3}{2}\,\mathrm{Li}_2(-2) + \frac{3}{2}(\ln 3)(\ln 2) + \frac{\pi^2}{8} + 3\bar{w}^{1/2} \ln \frac{3}{4} \\ & + \frac{3}{4}\,\bar{w} \left(3\,\mathrm{Li}_2(-2) + 3(\ln 3)(\ln 2) + 4 \ln 2 + \frac{\pi^2}{4} \right) \\ & + O(\bar{w}^{3/2}, \bar{w}^{3/2} \ln \bar{w}) \Bigg], \end{aligned} \tag{A.7}$$



$$\begin{aligned} = \frac{r_{0L}^{-3/2}}{(4\pi)^3} \Bigg[& c_2 + 2\bar{w}^{1/2}(\ln \bar{w} + 6 \ln 2 - 2) + \bar{w}(c_2 - 6 - 4 \ln 2) \\ & + O(\bar{w}^{3/2}, \bar{w}^{3/2} \ln \bar{w}) \Bigg], \end{aligned} \tag{A.8}$$

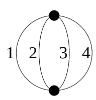
where c_2 given by Eq. (20).

For finite m_i^2 the diagrams (B)–(D) and (F) are finite in three dimensions at

infinite cutoff whereas (E), Eq. (A.5), has a single pole $\sim (d-3)^{-1}$. Analytic results for the diagrams (B)–(D) and (F), Eqs. (A.2)–(A.4) and (A.6), are given in Eqs. (A.24), (A.25) and (A.27) of Ref. [27] and in Eq. (5) of the Erratum [27].

Regarding the $d = 3$ divergent diagram (E), the $\overline{\mathrm{MS}}$ scheme was used in Ref. [27] which differs from our approach. Within our approach the result for diagram (E) near $d = 3$ is represented as

$$u_0^2$$



$$= \frac{u_0^{-1+d/\epsilon}}{(4\pi)^3} \sum_{i=1}^{4} m_i \Bigg[-\frac{1/2}{\epsilon - 1} - 2 + \frac{3}{4}\gamma - \frac{1}{4}\ln\left[16\pi^3\right] + \frac{1}{2}\ln\frac{m_i}{u_0} + \ln\frac{\sum_i m_i}{u_0} + O(\epsilon - 1)\Bigg] \qquad \text{(A.9)}$$

with $\gamma \equiv C_{\mathrm{Euler}}$ being Euler's constant and $\epsilon = 4-d$. The pole term $\sim (\epsilon-1)^{-1}$ is cancelled by a term of order $O(u_0^2)$ that arises from performing the mass shift $r_0 = r_0' + \delta r_0$ in the one-loop contribution of the Helmholtz free energy. Eq. (A.9) corresponds to Eqs. (21) and (A.26) of Ref. [27] and Eq. (1) of the Erratum [27]. Note that in our Eq. (A.9) there is no additional parameter corresponding to the $\overline{\mathrm{MS}}$ scale parameter $\bar{\mu}$ of Ref. [27].

B Correlation lengths in three-loop order

In this Appendix we derive Eqs. (33) and (23). Above T_c, the correlation length is defined via

$$\xi_+^2 = \mathring{\chi}_+(0)\, \partial \mathring{\chi}_+(q)^{-1} / \partial q^2 \Big|_{q=0} \qquad \text{(B.1)}$$

where $\mathring{\chi}_+(q)^{-1} = \Gamma_0^{(0,2)}(q, r_0, u_0)$ is the inverse susceptibility at finite wavenumber q. The two-point vertex function $\Gamma_0^{(0,2)}(q, r_0, u_0)$ is given by

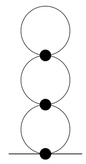
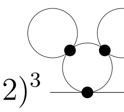
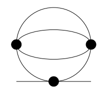
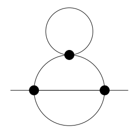
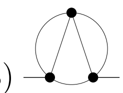
$$\Gamma_0^{(0,2)}(q, r_0, u_0) = r_0 + q^2 - \Sigma_0(q, r_0, u_0) \qquad \text{(B.2)}$$

where the self-energy

$$\Sigma_0(q, r_0, u_0) = \sum_{m=1}^{\infty} (-u_0)^m \Sigma_0^{(m)}(q, r_0) \qquad \text{(B.3)}$$

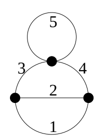
is the sum of all one-particle irreducible m-loop diagrams with two (amputated) external legs. In Eqs. (A.1)–(A.5) of Ref. [6] the functions $\Gamma_0^{(0,2)}(q, r_0, u_0)$,

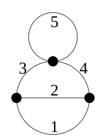
Eq. (B.2), and $\xi_+(r_0, u_0)$, Eq. (B.1), are given in their two-loop approximation. The diagrams of the three-loop contribution to Eq. (B.3) are given by

$$\Sigma_0^{(3)}(q, r_0) = 64(n+2)^3$$

$$+\, 64(n+2)^3$$

$$+\, 128(n+2)^2$$

$$+\, 384(n+2)^2$$

$$+\, 128(n+2)(n+8)$$

$$. \quad \text{(B.4)}$$

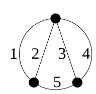
For $q = 0$ the lines denote the standard propagator $(r_0 + p^2)^{-1}$ above T_c. The derivative of $\Sigma_0^{(3)}(q, r_0)$ with respect to q^2 yields

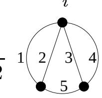
$$\frac{\partial}{\partial q^2} \Sigma_0^{(3)}(q, r_0)\bigg|_{q=0} =$$

$$-\, 384(n+2)^2 \left[\frac{\epsilon}{d} \frac{\partial}{\partial m_1^2} \right.$$


$$+\, \frac{2r_0}{d} \frac{\partial^2}{(\partial m_1^2)^2}$$


$$\left. \vphantom{\frac{\partial}{\partial}} \right]_{m_i^2 = r_0}$$

$$-\, 128(n+2)(n+8) \left[\frac{\epsilon}{d} \frac{\partial}{\partial m_5^2} \right.$$


$$+\, \frac{2r_0}{d} \frac{\partial^2}{(\partial m_5^2)^2}$$


$$\left. \vphantom{\frac{\partial}{\partial}} \right]_{m_i^2 = r_0} . \quad \text{(B.5)}$$

All diagrams appearing in Eqs. (B.4) and (B.5) can be calculated (at $d = 3$ and $q = 0$) using Appendix A with $m_i^2 \equiv r_0$.

Following Ref. [6] we invert $\xi_+(r_0, u_0)$, Eq. (B.1), and get the function $r_0(\xi_+, u_0)$. Subtracting $\delta r_0 = r_0 - r_0'$, Eq. (6), from $r_0(\xi_+, u_0)$ and letting $d \to 3$ we obtain Eq. (33). Below T_c we get $r_0'(\xi_-, u_0)$, Eq. (23), at $d = 3$ as described in Ref. [6].

References

[1] J. Goldstone, Nuovo Cimento 19 (1961) 154.

[2] H. Wagner, Z. Phys. 195 (1966) 273.

[3] A.Z. Patashinskii and V.L. Pokrovskii, Zh. Eksp. Teor. Fiz. 64 (1973) 1445 [Sov. Phys. JETP 37 (1973) 733]; E. Brézin, D.J. Wallace, and K.G. Wilson, Phys. Rev. B 7 (1973) 232; E. Brézin and D.J. Wallace, Phys. Rev. B 7 (1973) 1967; D.R. Nelson, Phys. Rev. B 13 (1976) 2222; L. Schäfer and H. Horner, Z. Phys. B 29 (1978) 251; V.L. Pokrovskii, Adv. Phys. 28 (1978) 595; I.D. Lawrie, J. Phys. A 14 (1981) 4576.

[4] C. Bervillier, Phys. Rev. B 14 (1976) 4964; ibid. B 34 (1986) 8141; M.C. Chang and A. Houghton, Phys. Rev. B 21 (1980) 1881; Y. Okabe and K. Ideura, Prog. Theor. Phys. 66 (1981) 8141; N.A. Shpot, Zh. Eksp. Teor. Fiz. 98 (1990) 1762 [Sov. Phys. JETP 71 (1990) 989].

[5] R. Schloms and V. Dohm, Phys. Rev. B 42 (1990) 6142; ibid. 46 (1992) 5883 (E).

[6] S.S.C. Burnett, M. Strösser, and V. Dohm, Nucl. Phys. B 504 [FS] (1997) 665.

[7] C. Bagnuls, C. Bervillier, D.I. Meiron, and B.G. Nickel, Phys. Rev. B 35 (1987) 3585.

[8] F.J. Halfkann and V. Dohm, Z. Phys. B 89 (1992) 79.

[9] R. Guida, J. Zinn-Justin, Nucl. Phys. B 489 [FS] (1997) 626.

[10] S.A. Larin, M. Mönnigmann, M. Strösser, and V. Dohm, Phys. Rev. B 58 (1998) 3394.

[11] V. Dohm, in *Proceedings of the 1997 NASA/JPL Microgravity Fundamental Physics Workshop*, Santa Barbara, May 7-9, 1997, NASA Document D-15677 (1998), p. 189; DRL (DARA) Proposal, 50 WM 9669 (1997); cond-mat/9803095.

[12] J.A. Lipa, V. Dohm, U.E. Israelsson, and M.J. DiPirro, NASA Proposal, NRA 94-OLMSA-05 (1995).

[13] See, e.g., V. Privman, P.C. Hohenberg and A. Aharony, in *Phase Transitions and Critical Phenomena*, ed. C. Domb and J.L. Lebowitz (Academic Press, London, 1991), Vol. 14, p. 1; and references therein.

[14] G. Ahlers, Phys. Rev. A 3, 696 (1971); ibid. A 8 (1973) 530; D. Greywall and G. Ahlers, Phys. Rev. Lett. 28 (1972) 1251; Phys. Rev. A 7 (1973) 2145; K.H. Mueller, G. Ahlers, and F. Pobell, Phys. Rev. B 14 (1976) 2096; G. Ahlers, in: *Quantum Liquids*, ed. J. Ruvalds and T. Regge (North Holland, Amsterdam, 1978); J.A. Lipa and T.C.P. Chui, Phys. Rev. Lett. 51 (1983) 2291; A. Singsaas and G. Ahlers, Phys. Rev. B 30 (1984) 5103.

[15] J.A. Lipa, D.R. Swanson, J.A. Nissen, T.C.P. Chui, and U.E. Israelsson, Phys. Rev. Lett. 76 (1996) 944.

[16] V. Dohm, Phys. Rev. Lett. 53 (1984) 1379; 2520; in: *Proceedings of the* 7^{th} *International Conference on Low Temperature Physics*, ed. U. Eckern, A. Schmidt, W. Weber, and H. Wühl, (North Holland, Amsterdam, 1984) 953; in: *Applications of Field Theory to Statistical Mechanics*, ed. L. Garrido (Springer, Berlin, 1985) 263; R. Schloms, J. Eggers, and V. Dohm, Jpn. J. Appl. Phys. Suppl. 26-3 (1987) 49.

[17] V. Dohm, Z. Phys. B 60 (1985) 61.

[18] V. Dohm, J. Low Temp. Phys. 69 (1987) 51.

[19] F. Wegner, Phys. Rev. B 5 (1972) 4529.

[20] R. Schloms and V. Dohm, Nucl. Phys. B 328 (1989) 639.

[21] G. Parisi, in *Proceedings of the 1973 Cargèse Summer Institute*; J. Stat. Phys. 23 (1980) 49.

[22] G.A. Baker, B.G. Nickel, M.S. Green and D.I. Meiron, Phys. Rev. Lett. 36 (1976) 1351; B.G. Nickel, D.I. Meiron and G.B. Baker, Univ. of Guelph Report (1977); G.A. Baker, B.G. Nickel and D.I. Meiron, Phys. Rev. B 17 (1978) 1365.

[23] C. Bagnuls and C. Bervillier, Phys. Rev. B 32 (1985) 7209.

[24] G. Münster and J. Heitger, Nucl. Phys. B 424 (1994) 582; C. Gutsfeld, J. Küster and G. Münster, Nucl. Phys. B 479 (1996) 654.

[25] V. Dohm, Phys. Rev. B 44 (1991) 2697.

[26] A. Esser, V. Dohm and X.S. Chen, Physica A 222 (1995) 355; X.S. Chen, V. Dohm and N. Schultka, Phys. Rev. Lett. 77 (1996) 3641.

[27] A.R. Rajantie, Nucl. Phys. B 480 (1996) 729; ibid. B 513 (1998) 761 (E).

[28] J. Zinn-Justin, Phys. Rep. 70 (1981) 109.

[29] A. Aharony and P. C. Hohenberg, Phys. Rev. B 13 (1976) 3081.

[30] V. Dohm, Physica Scripta T**49** (1993) 46.

[31] C. Bagnuls and C. Bervillier, Phys. Lett. A **195** (1994) 163.

[32] M. A. Anisimov, A. A. Povodyrev, V. D. Kulikov and J. V. Sengers, Phys. Rev. Lett. **75** (1995) 3146.

[33] K. Symanzik, Lett. Nuovo Cimento 8 (1973) 771.

[34] F. David, Commun. Math. Phys. 81 (1981) 149; S. Elitzur, Nucl. Phys. B 212 (1983) 501; I.D. Lawrie, J. Phys. A 18 (1985) 1141.

[35] E. Brézin, J.C. Le Guillou and J. Zinn-Justin, in: *Phase Transitions and Critical Phenomena*, ed. C. Domb and M.S. Green (Academic Press, New York, 1976) vol. 6, 125.

[36] G. 't Hooft and M. Veltman, Nucl. Phys. B 44 (1972) 189; G. 't Hooft, Nucl. Phys. B 61 (1973) 455.

[37] V. Dohm, Z. Phys. B 61 (1985) 193.

[38] H.J. Krause, R. Schloms and V. Dohm, Z. Phys. B 79 (1990) 287; ibid. 80 (1990) 313 (E).

[39] S.S.C. Burnett, private comunication.

[40] M. Mönnigmann, Diplom Thesis, RWTH Aachen (1998).

[41] R. Schloms and V. Dohm, Europhys. Lett. 3 (1987) 413.

[42] See also Appendix C of Ref. [7] where the ϵ-expansion result for A^+/A^- is given up to $O(\epsilon^2)$ for general n without resummation. Setting $\epsilon = 1$ in this expression would yield $A^+/A^- = 0.7815$ for $n = 2$.

[43] J.C. Le Guillou and J. Zinn-Justin, Phys. Rev. B 21 (1980) 3976.

[44] S. Milošević and H.E. Stanley, Phys. Rev. B 6 (1972) 986; B 6 (1972) 1002.